

# Sharp transition in self-avoiding walk on random conductors on a tree

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## Abstract

We consider self-avoiding walk on a tree with random conductances. It is proven that in the weak disorder regime, the quenched critical point is equal to the annealed one, and that in the strong disorder regime, these critical points are strictly different. Derrida and Spohn, and Baffet, Patrick and Pulé give the exact value of the quenched critical point. We give another heuristic approach by the fractional moment estimate.

## 1 Introduction and the main theorem

Self-avoiding walk (SAW) is a statistical-mechanical model that has been studied in both physics school and mathematics school. We have currently considered SAW in a random medium. The model we treat in this paper is SAW on a tree with random conductors, which can be regarded as a directed polymer model on a disordered tree. We consider a SAW  $\omega$  on a degree- $\ell$  tree  $\mathbb{T}^\ell$ . We denote by  $|\omega|$  the length of  $\omega$  and by  $\Omega(x; n)$  the set of SAWs of length  $n$  from  $x \in \mathbb{T}^\ell$ . We also denote by  $\mathbb{B}^\ell$  the set of nearest-neighbor bonds on  $\mathbb{T}^\ell$ , and we define the set of random conductors  $\mathbf{X} = \{X_b\}_{b \in \mathbb{B}^\ell}$  as a collection of i.i.d. random variables whose probability law is denoted by  $\mathbb{P}$ . We set some notations that are common in the study of SAW: the number of  $n$ -step SAWs  $c_n$  and the connective constant  $\mu = \lim_{n \rightarrow \infty} c_n^{1/n}$  (due to the subadditivity of SAW, the existence of this limit is guaranteed). Note that  $c_n = \ell(\ell - 1)^{n-1}$  and  $\mu = \ell - 1$  on  $\mathbb{T}^\ell$ .

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Given the energy cost  $h \in \mathbb{R}$  and the strength of randomness  $\beta \geq 0$ , we define the quenched susceptibility at  $x \in \mathbb{T}^\ell$  by

$$\hat{\chi}_{h,\beta,\mathbf{X}}(x) = \sum_{\omega \in \Omega(x)} e^{-\sum_{j=1}^{|\omega|} (h + \beta X_{b_j})}, \quad (1.1)$$

where  $b_j \equiv b_j(\omega) = (\omega_{j-1}, \omega_j)$ . Since  $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$  is monotonic in  $h$ , we can define the quenched critical point by

$$\hat{h}_{\beta,\mathbf{X}}^q(x) = \inf\{h \in \mathbb{R} : \hat{\chi}_{h,\beta,\mathbf{X}}(x) < \infty\}. \quad (1.2)$$

In [3], we prove on  $\mathbb{Z}^d$  that  $\hat{h}_{\beta,\mathbf{X}}^q(x)$  is independent of the reference point  $x$  and it is a degenerate random variable. Moreover, it is valid for the case that  $\{X_b\}$  is a collection of integrable random variables whose law  $\mathbb{P}$  is translation-invariant and ergodic. Henceforth, we simply write the quenched critical point by  $\hat{h}_\beta^q$ .

In the study of the disordered systems, it is standard to investigate the annealed model. The annealed observables are easy to compute in most cases since we can reduce the annealed model to a homogeneous one. By virtue of the self-avoidance constraint on  $\omega$  and the i.i.d. property of  $\mathbf{X}$ , we can directly compute the annealed susceptibility  $\mathbb{E}[\hat{\chi}_{h,\beta,\mathbf{X}}(x)]$  as

$$\mathbb{E}[\hat{\chi}_{h,\beta,\mathbf{X}}(x)] = \sum_{n=0}^{\infty} c_n \lambda_\beta^n e^{-hn} = \chi_{h-\log \lambda_\beta}, \quad (1.3)$$

where  $\lambda_\beta$  is the Laplace transform of the distribution  $\mathbb{P}$ , i.e.,  $\lambda_\beta = \mathbb{E}[e^{-\beta X_b}]$ . Let

$$h_\beta^a = \log \mu + \log \lambda_\beta, \quad (1.4)$$

then  $\mathbb{E}[\hat{\chi}_{h,\beta,\mathbf{X}}(x)] < \infty$  if and only if  $h > h_\beta^a$ . Thus  $h_\beta^a$  is called the annealed critical point.

According to classical theorems by Kahane and Peyrière [5] and Beggins [2], it is known that there exists a transition behavior in a directed polymer model on a disordered tree. Let

$$Z_n = \frac{1}{c_n} \sum_{\omega \in \Omega(x;n)} e^{-\sum_{j=1}^n (\beta X_{b_j} + \log \lambda_\beta)}, \quad (1.5)$$

then the susceptibility  $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$  is represented as

$$\hat{\chi}_{h,\beta,\mathbf{X}}(x) = \sum_{n=0}^{\infty} c_n \lambda_\beta^n e^{-hn} Z_n. \quad (1.6)$$

For  $x \in \mathbb{T}^\ell$ , Let  $\mathcal{F}_n(x) = \sigma(X_b : b = (u, v) \in \mathbb{B}^\ell, |u - x| \leq n, |v - x| \leq n)$ , then  $Z_n$  is a positive martingale with respect to  $\mathcal{F}_n(x)$ . By applying the martingale convergence theorem and Kolmogorov's 0-1 law, there exists a non-negative random variable  $Z_\infty := \lim_{n \rightarrow \infty} Z_n$  and the probability  $\mathbb{P}(Z_\infty = 0)$  is equal to either 0 or 1. For  $\beta \geq 0$ , we define the function

$$f(\beta) = h_\beta^a - \beta \left( \frac{d}{d\beta} h_\beta^a \right). \quad (1.7)$$

Since  $\frac{d}{d\beta} f(\beta)$  is negative for  $\beta > 0$  and  $f(0) = \log(\ell - 1) > 0$  for  $\ell \geq 3$ , the function  $f(\beta)$  is decreasing in  $\beta > 0$  and there exists some  $\beta_c$  such that  $f(\beta_c) = 0$ . Kahane and Peyrière [5] and Beggs [2] show that

$$\begin{aligned} \mathbb{P}(Z_\infty > 0) &= 1 \Leftrightarrow \beta < \beta_c \ (f(\beta) > 0), \\ \mathbb{P}(Z_\infty = 0) &= 1 \Leftrightarrow \beta \geq \beta_c \ (f(\beta) \leq 0). \end{aligned} \quad (1.8)$$

For  $\beta < \beta_c$ , we call the weak disorder regime, and for  $\beta > \beta_c$ , the strong disorder regime. Derrida and Spohn [4] prove that the quenched critical point

$$\hat{h}_\beta^q = \begin{cases} h_\beta^a & \text{if } \beta \leq \beta_c, \\ \frac{\beta}{\beta_c} h_{\beta_c}^a & \text{if } \beta > \beta_c, \end{cases} \quad (1.9)$$

Buffet, Patrick and Pulé [1] also prove that  $\hat{h}_\beta^q = \frac{\beta}{\beta_c} h_{\beta_c}^a$  by applying the martingale argument. The following is the main theorem of this paper.

**Theorem 1.1.** *For  $\ell \geq 3$ , in (1.9) the critical parameter  $\beta_c$  is given by  $\theta_c \beta$  where  $\theta_c$  is the value that minimizes the function  $\log r(\theta)$ , where  $r(\theta)$  is defined by*

$$r(\theta) = (\ell - 1) \mathbb{E} \left[ \left( \frac{e^{-\beta X_b}}{(\ell - 1) \lambda_\beta} \right)^\theta \right]. \quad (1.10)$$

Note that the case  $\ell = 2$  is equivalent to the case  $\mathbb{Z}^1$ . Since  $c_n = 2$  and two SAW paths are independent on  $\mathbb{Z}^1$ , it can be proven that  $\hat{h}_\beta^q = -\beta \mathbb{E}[X_b]$  on  $\mathbb{Z}$  by the individual ergodic theorem (the strong law of large numbers if i.i.d. case). On  $\mathbb{Z}^{d \geq 2}$ , however, since  $c_n$  grows exponentially, it is hard to control the speed of convergence along the SAWs at the same time. Because of the entropic effect, we strongly believe that  $\log \mu - \beta \mathbb{E}[X_b] < \hat{h}_\beta^q$ . Therefore, the exact value of quenched critical point on  $\mathbb{Z}^{d \geq 2}$  remains an open problem.

## 2 In the weak disorder regime

As an immediate consequence from (1.8) and (1.9), we can show that for  $\ell \geq 3$ , the critical exponent is almost surely equal to 1. We consider the

quenched susceptibility at  $h = h_\beta^a + \delta$  for any  $\beta \in [0, \beta_c)$  and  $\delta > 0$ . Since  $Z_n$  converges to  $Z_\infty$  as  $n \rightarrow \infty$ ,  $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$  is bounded from above as

$$\begin{aligned}\hat{\chi}_{h,\beta,\mathbf{X}} &\leq \sum_{n=0}^{N-1} \frac{c_n}{\mu^n} e^{-\delta n} Z_n + \sum_{n=N}^{\infty} \frac{c_n}{\mu^n} e^{-\delta n} (Z_\infty + \varepsilon), \\ &\leq \frac{\ell N}{\ell - 1} \left( \max_{0 \leq n \leq N-1} Z_n \right) + \frac{\ell(Z_\infty + \varepsilon)}{(\ell - 1) e^{\delta N}} \frac{1}{1 - e^{-\delta}}.\end{aligned}\quad (2.1)$$

and is also bounded from below as

$$\hat{\chi}_{h,\beta,\mathbf{X}} \geq \sum_{n=N}^{\infty} \frac{c_n}{\mu^n} e^{-\delta n} (Z_\infty - \varepsilon) = \frac{\ell(Z_\infty - \varepsilon)}{(\ell - 1) e^{\delta N}} \frac{1}{1 - e^{-\delta}}. \quad (2.2)$$

By (2.1) and (2.2), there exist random variables  $0 < c < C < \infty$  depending on  $\omega$ ,  $\mathbf{X}$  and  $\varepsilon$  such that

$$\frac{c}{h - h_\beta^a} \leq \hat{\chi}_{h,\beta,\mathbf{X}}(x) \leq \frac{C}{h - h_\beta^a}, \quad \text{as } h \downarrow h_\beta^a. \quad (2.3)$$

### 3 In the strong disorder regime

#### 3.1 The upper bound

For  $\ell \geq 3$ , the quenched critical point  $\hat{h}_\beta^q$  is almost surely smaller than  $\frac{\beta}{\beta_c} h_{\beta_c}^a$  in the strong disorder regime. To prove this, we estimate the rate of convergence of  $Z_n$ . In this section, we denote by  $Z_n^{(x)}$  to emphasize the starting point  $x$ . We introduce another martingale defined by

$$\tilde{Z}_n^{(y)} = \frac{1}{(\ell - 1)^n} \sum_{\eta \in \tilde{\Omega}(y; n)} e^{-\sum_{j=1}^n (\beta X_{b_j(\eta)} + \log \lambda_\beta)}, \quad (3.1)$$

where  $\tilde{\Omega}(y; n) = \{\omega = (\omega_0, \dots, \omega_n) \in \Omega(y; n) : \forall j, \omega_j \neq x\}$  is the set of SAWs on a forward tree for  $y$  neighboring to  $x$ . By subadditivity of SAW,

$$Z_n^{(x)} \leq \sum_{\substack{y \in \mathbb{T}^\ell \\ |x-y|=1}} \frac{e^{-\beta X_{(x,y)}}}{\ell \lambda_\beta} \tilde{Z}_{n-1}^{(y)}, \quad \tilde{Z}_{n-1}^{(y)} \leq \sum_{\substack{z \in \mathbb{T}^\ell \setminus \{x\} \\ |y-z|=1}} \frac{e^{-\beta X_{(y,z)}}}{(\ell - 1) \lambda_\beta} \tilde{Z}_{n-2}^{(z)}. \quad (3.2)$$

Due to the transitivity of a homogeneous degree tree and the i.i.d. property of  $\mathbf{X}$ , we obtain

$$\mathbb{E}[Z_n^\theta] \leq \sum_{\substack{y \in \mathbb{T}^\ell \\ |x-y|=1}} \mathbb{E}\left[\left(\frac{e^{-\beta X(x,y)}}{\ell \lambda_\beta}\right)^\theta\right] \mathbb{E}[\tilde{Z}_{n-1}^\theta] \leq \ell^{1-\theta} \frac{\lambda_{\theta\beta}}{\lambda_\beta^\theta} \mathbb{E}[\tilde{Z}_{n-1}^\theta], \quad (3.3)$$

$$\begin{aligned} \mathbb{E}[\tilde{Z}_{n-1}^\theta] &\leq \sum_{\substack{z_1 \in \tilde{\mathbb{T}}^\ell \\ |y-z_1|=1}} \mathbb{E}\left[\left(\frac{e^{-\beta X(y,z_1)}}{(\ell-1)\lambda_\beta}\right)^\theta\right] \mathbb{E}[\tilde{Z}_{n-2}^\theta] \leq (\ell-1)^{1-\theta} \frac{\lambda_{\theta\beta}}{\lambda_\beta^\theta} \mathbb{E}[\tilde{Z}_{n-2}^\theta] \\ &\leq \dots \leq \left\{(\ell-1)^{1-\theta} \frac{\lambda_{\theta\beta}}{\lambda_\beta^\theta}\right\}^{n-1}. \end{aligned} \quad (3.4)$$

Substituting (3.4) into (3.3), we have

$$\mathbb{E}[Z_n^\theta] \leq \left(\frac{\ell}{\ell-1}\right)^{1-\theta} r(\theta)^n, \quad (3.5)$$

where  $r(\theta)$  is defined by (1.10). Therefore, by the definition of the annealed critical point  $h_\beta^a$ , we have

$$\log r(\theta) = h_{\theta\beta}^a - \theta h_\beta^a. \quad (3.6)$$

We will show that  $\mathbb{E}[Z_n^\theta]$  decays exponentially. We compute the first and second derivatives of  $\log r(\theta)$ .

$$\frac{d}{d\theta}(\log r(\theta)) = -\beta \frac{\mathbb{E}[X_b e^{-\theta\beta X_b}]}{\lambda_{\theta\beta}} - h_\beta^a = \beta \left( \frac{d}{d\beta} h_\beta^a \Big|_{\beta=\theta\beta} \right) - h_\beta^a, \quad (3.7)$$

$$\frac{d^2}{d\theta^2}(\log r(\theta)) = \beta^2 \left\{ \frac{\mathbb{E}[X_b^2 e^{-\theta\beta X_b}]}{\lambda_{\theta\beta}} - \left( \frac{\mathbb{E}[X_b e^{-\theta\beta X_b}]}{\lambda_{\theta\beta}} \right)^2 \right\} \geq 0. \quad (3.8)$$

Thus, we can say that  $\log r(\theta)$  is convex. Since

$$\frac{d}{d\theta}(\log r(1)) = \beta \left( \frac{d}{d\beta} h_\beta^a \right) - h_\beta^a = -f(\beta) > 0 \quad (3.9)$$

by (3.7),  $\log r(0) = \log(\ell-1) > 0$  and  $\log r(1) = 0$  (see Figure 1), in the strong disorder regime, we conclude that  $\mathbb{E}[Z_n^\theta]$  is exponentially decaying in the strong disorder regime.

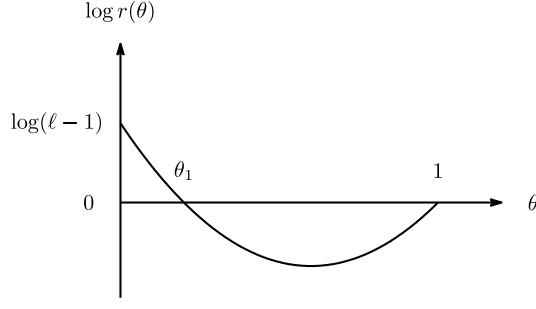


Figure 1: For  $\theta \in (\vartheta_1, 1)$ ,  $\log r(\theta)$  is strictly negative.

For  $h = h_\beta^a - \frac{1}{\theta} \log \frac{1}{r(\theta)} + \delta$  and  $\delta > 0$ ,

$$\hat{\chi}_{h,\beta,\mathbf{X}}(x) = \frac{\ell}{\ell-1} \sum_{n=0}^{\infty} e^{-\delta n} r(\theta)^{-n/\theta} Z_n. \quad (3.10)$$

For any  $\varepsilon > 0$ , by Markov's inequality,

$$\mathbb{P}(Z_n \geq (r(\theta) + \varepsilon)^{n/\theta}) \leq \frac{\mathbb{E}[Z_n^\theta]}{(r(\theta) + \varepsilon)^n} \leq \left(\frac{\ell}{\ell-1}\right)^{1-\theta} \left(\frac{r(\theta)}{r(\theta) + \varepsilon}\right)^n. \quad (3.11)$$

Then, by the Borel-Cantelli lemma, the event  $\{Z_n < (r(\theta) + \varepsilon)^{n/\theta}\}$  occurs for all but for finitely many  $n$ . We can control  $\varepsilon > 0$  depending on  $\delta > 0$  for the summation in (3.10) to be finite.

$$e^{-\delta n} r(\theta)^{-n/\theta} Z_n \leq \exp \left\{ -n \left( \delta - \frac{1}{\theta} \log \left( 1 + \frac{\varepsilon}{r(\theta)} \right) \right) \right\}, \quad (3.12)$$

so that  $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$  is almost surely finite if we choose  $\varepsilon \leq r(\theta)e^{\theta\delta}$ . This implies that for any  $\theta \in (\theta_1, 1)$ ,

$$\hat{h}_\beta^q \leq h_\beta^a - \frac{1}{\theta} \log \frac{1}{r(\theta)}. \quad (3.13)$$

To optimize an upper bound (3.13), we compute a derivative of  $\frac{1}{\theta} \log r(\theta)$ .

$$\frac{\theta}{d\theta} \left( \frac{1}{\theta} \log r(\theta) \right) = -\frac{1}{\theta^2} \left\{ h_{\theta\beta}^a - \theta\beta \left( \frac{d}{d\beta} h_\beta^a \Big|_{\beta=\theta\beta} \right) \right\} = -\frac{1}{\theta^2} f(\theta\beta). \quad (3.14)$$

Therefore,

$$\frac{\theta}{d\theta} \left( \frac{1}{\theta} \log r(\theta) \right) \begin{cases} < 0 & \text{if } \theta\beta < \beta_c, \\ = 0 & \text{if } \theta\beta = \beta_c, \\ > 0 & \text{if } \theta\beta > \beta_c. \end{cases} \quad (3.15)$$

For  $\theta_c = \frac{\beta_c}{\beta}$ , we have the upper bound on the quenched critical point.

$$\hat{h}_\beta^q \leq h_\beta^a - \frac{1}{\theta_c} \log \frac{1}{r(\theta_c)} = \frac{\beta}{\beta_c} h_{\beta_c}^a. \quad (3.16)$$

### 3.2 The lower bound

To prove that  $\hat{h}_\beta^a = \frac{\beta}{\beta_c} h_{\beta_c}^a$ , we need to show that for  $\ell \geq 3$ ,  $\hat{h}_\beta^a$  is almost surely larger than  $\frac{\beta}{\beta_c} h_{\beta_c}^a$  in the strong disorder regime. First, for arbitrary  $\varepsilon > 0$ , we define the event  $A_{n,\varepsilon}$ ,

$$A_{n,\varepsilon} = \{Z_n > (r(\theta_c) - \varepsilon)^{n/\theta_c}\}. \quad (3.17)$$

Then, we have

$$\begin{aligned} \mathbb{P}(\hat{\chi}_{h,\beta,\mathbf{x}}(x) = \infty) &= \underbrace{\mathbb{P}(\hat{\chi}_{h,\beta,\mathbf{x}}(x) = \infty \mid \limsup_{n \rightarrow \infty} A_{n,\varepsilon})}_{=1} \mathbb{P}(\limsup_{n \rightarrow \infty} A_{n,\varepsilon}) \\ &\geq \lim_{n \rightarrow \infty} \mathbb{P}(A_{n,\varepsilon}). \end{aligned} \quad (3.18)$$

The event  $\{\hat{\chi}_{h,\beta,\mathbf{x}}(x) = \infty\}$  is translation-invariant. Since  $\mathbb{P}$  is ergodic,  $\mathbb{P}(\hat{\chi}_{h,\beta,\mathbf{x}}(x) = \infty)$  is either zero or one. Therefore, it suffices to show that the rightmost limit in (3.18) is positive. By the Cauchy-Schwarz inequality,

$$\begin{aligned} 1 = \mathbb{E}[Z_n] &= \mathbb{E}[Z_n \mathbf{1}_{\{A_{n,\varepsilon}\}}] + \mathbb{E}[Z_n \mathbf{1}_{\{A_{n,\varepsilon}^c\}}] \\ &\leq \mathbb{E}[Z_n^2]^{1/2} \mathbb{P}(A_{n,\varepsilon})^{1/2} + (r(\theta_c) - \varepsilon)^{n/\theta_c} (1 - \mathbb{P}(A_{n,\varepsilon})). \end{aligned} \quad (3.19)$$

Letting  $\mathbb{E}[Z_n^2]^{1/2} =: \sigma$ ,  $\mathbb{P}(A_{n,\varepsilon})^{1/2} =: a$  and  $(r(\theta_c) - \varepsilon)^{n/\theta_c} =: R_n$ , we obtain

$$g(a) := R_n a^2 - \sigma a + 1 - R_n \leq 0, \quad (3.20)$$

and  $g(0) = 1 - R_n \geq 0$  and  $g(1) = 1 - \sigma$ . By Lemma 3.1 below,  $g(1)$  is negative for  $n$  large enough. We have also known  $R_n$  is small for  $n$  large enough, so that we can say that  $g(0) = 1 - R_n$  is positive. Therefore, there exists  $a_0$  such that  $g(a_0) = 0$  and for  $n$  large enough, we can say that (3.20) implies  $\mathbb{P}(A_{n,\varepsilon}) > 0$ . Hence we conclude that  $\hat{h}_\beta^a = \frac{\beta}{\beta_c} h_{\beta_c}^a$  almost surely.

**Lemma 3.1.** *The second moment  $\mathbb{E}[Z_n^2]$  diverges as  $n \rightarrow \infty$ .*

*Proof of Lemma 3.1.* We compute  $\mathbb{E}[Z_n^2]$ . By the definition of  $Z_n$ ,

$$\mathbb{E}[Z_n^2] = \mathbb{E}\left[\frac{1}{c_n^2 \lambda_\beta^{2n}} \sum_{\omega \in \Omega(x;n)} \sum_{\eta \in \Omega(x;n)} \prod_{j=1}^n e^{-\beta(X_{b_j(\omega)} + X_{b_j(\eta)})}\right]. \quad (3.21)$$

Recall that  $c_n = \ell(\ell - 1)^{n-1}$ . Due to the property of the tree graph (see Figure 2), for fixed  $\omega$ ,

$$\begin{aligned} &\sum_{\eta \in \Omega(x;n)} \mathbb{E}\left[\prod_{j=1}^n e^{-\beta(X_{b_j(\omega)} + X_{b_j(\eta)})}\right] \\ &= (\ell - 1)^n \lambda_\beta^n \lambda_\beta^n + \frac{\ell - 2}{\ell - 1} \sum_{k=1}^{n-1} (\ell - 1)^{n-k} \lambda_{2\beta}^k (\lambda_\beta \lambda_\beta)^{n-k} + \lambda_{2\beta}^n, \end{aligned} \quad (3.22)$$

where the first part implies that  $\eta$  has no common edges with  $\omega$ , the second part implies that  $\eta$  has  $k$  common edges with  $\omega$ , and the last part implies that  $\eta$  coincides with  $\omega$ .

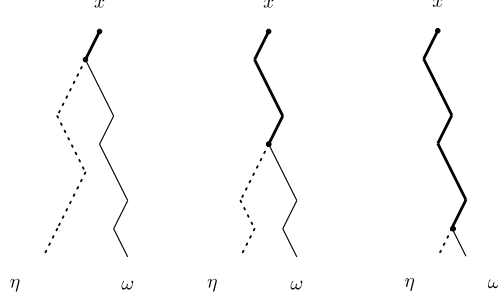


Figure 2: The bold edges present the common edges of  $n$ -step SAWs  $\omega$  and  $\eta$ . The rest part of  $\eta$  (dotted) is independent of the rest part of  $\omega$ .

Substituting (3.22) into (3.21), we obtain

$$\begin{aligned}\mathbb{E}[Z_n^2] &= \left(\frac{\ell-1}{\ell}\right) \left\{ \frac{\ell-2}{\ell} \sum_{k=1}^{n-1} \left( (\ell-1) \frac{\lambda_{2\beta}}{(\ell-1)^2 \lambda_\beta^2} \right)^k + \left( (\ell-1) \frac{\lambda_{2\beta}}{(\ell-1)^2 \lambda_\beta^2} \right)^n \right\} \\ &= \left(\frac{\ell-1}{\ell}\right) \left\{ \frac{\ell-2}{\ell} \sum_{k=1}^{n-1} r(2)^k + r(2)^n \right\},\end{aligned}\tag{3.23}$$

From the property of  $\log r(\theta)$ , we know  $\log r(1) = 0$  and  $\log r(2) > 0$ . Therefore, as  $n \rightarrow \infty$ ,  $\mathbb{E}[Z_n^2]$  diverges.  $\blacksquare$

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## References

- [1] E. Buffet, A. Patrick, and J. V. Pulé. Directed polymers on trees: a martingale approach. *J. Phys. A*. **26** (1993): 1823–1834.
- [2] J. D. Biggins. Martingale convergence in the branching random walk. *J. Appl. Probab.* **14** (1977): 25–37.
- [3] Y. Chino, and A. Sakai. The quenched critical point for self-avoiding walk on random conductors. *J. Stat. Phys.* **163** (2016): 754–764.
- [4] B. Drrida, and H. Spohn. Polymers on disordered trees, spin glasses, and travelling waves. *J. Stat. Phys.* **51** (1988): 817–840.
- [5] J. -P. Kahane, and J. Peyrière. Sur certaines martingales de Benoit Mandelbrot. *Advances in Math.* **22** (1976): 131–145.